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Infinite Number of Changes of Sign for A Difference of Two Number-Theoretic Functions

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Abstract

In this paper, we present two number-theoretic functions F and G for which $p_r F(n) - p_{r-1} G(n)$ is both positive and negative infinitely often, where n has at least k distinct prime factors $(k \geq 1)$ and (p_{r-1}, p_r) is a couple of two consecutive primes. To be precise, we will construct infinite sequences $(n_i)_{i\geq 1}, (m_i)_{i\geq 1}$ such that,

$$\frac{F(n_i)}{G(n_i)} > \frac{p_{r-1}}{p_r} > \frac{F(m_i)}{G(m_i)}, \text{ for } i = 1, 2, \ldots,$$

where each n_i and m_i has k distinct prime factors and $F(t)$ and $G(t)$ are either the Kernel or the Euler's function of the positive integer t.

Keywords: arithmetic functions; changes sign infinitely often; prime numbers; nonstandard analysis.

1 Introduction

Let $\gamma(n)$ and $\varphi(n)$ be the Kernel and the Euler's function of the positive integer n, respectively. Recall that if n has the prime factorization $n = q_1^{a_1} q_2^{a_2} \dots q_k^{a_k}$ with distinct primes $q_1, q_2, \dots, q_k$ and positive, integers $a_1, a_2, \dots, a_k$, then,

$$\gamma(n) = q_1 q_2 \dots q_k,$$

and

$$\varphi(n) = q_1^{a_1-1}(q_1-1)\, q_2^{a_2-1}(q_2-1) \dots q_k^{a_k-1}(q_k-1).$$

There are many questions in the literature dealing with diophantine equations and inequalities involving number-theoretic functions as well as the Euler's function and other multiplicative functions. For example, in [9, p. 99], Erdös asked to prove that $\varphi(n) > \varphi(n-\varphi(n))$ for almost all n, but that $\varphi(n) < \varphi(n-\varphi(n))$ for infinitely many n. That is, $\varphi(n) - \varphi(n-\varphi(n))$ may change signinfinitely often. Also, there are many papers on infinitely many signchanges. For example, it is shown that $\sigma(\varphi(n)) - \varphi(\sigma(n))$ is positive and negative infinitely often, see also [9, p. 99], where $\sigma(n)$ computes the sum of the positive divisors of n.

Let m and a be relatively prime positive integers. We denote by $\pi(x; m, a)$ the number of prime numbers $p \leq x$ such that $p \equiv a (\mathrm{mod}\ m)$. In the case $m = 4$, Littlewood [10, 14] proved that $\pi(x,4,1) - \pi(x,4,3)$ changes sign infinitely often. Further, in view of the papers [2, 3], significant works have been done on the sign changes of the Liouville function on quadratics and sign changes in sums of the Liouville function, respectively. Indeed, the authors proved that the binary sequence $\{\lambda(an^2+bn+c)\}_{n \geq A_0}$ with $a \in \mathbb{N}$ and $b, c \in \mathbb{Z}$ is either constant or it changes sign infinitely often, where A_0 is a fixed integer depending only on a, b and c, and λ denotes the Liouville function.

Recently, Rishabh and Chakraborty [1] studied similar problems on the sign changes of certain arithmetical functions at prime powers. In the same context, consider the difference,

$$F(n) - \alpha \cdot G(n), n \geq 1, \tag{1}$$

where F, G are two number-theoretic functions and $\alpha \in \mathbb{R}$ is understood as a parameter. Throughout this paper, $\alpha = \alpha(r) = p_{r-1}/p_r$, where p_r denotes the r-th prime number. Our purpose in this paper is to provide an improvement on a recent result [7] of the first author regarding whether (1) changes sign infinitely often or not. The primary goal of this work is summarized in the following two points. Indeed, we give a rational approximation to the parameter α. In the papers [7, 11], the authors studied the application of nonstandard analysis in the field arithmetic functions by interpreting large as *unlimited* (infinite, nonstandard) and close to as *having a limited* (finite, standard) *difference*. In the present work, we will apply some ideas from these papers where we use positive integers having unlimited number of distinct prime factors or unlimited prime powers, i.e., unlimited positive integers power of a single prime number.

Let $n \in \mathbb{N}$. We denote by $\omega(n)$ the number of distinct prime factors of n. For any $k \in \mathbb{N}$, define the subset W_k by,

$$W_k = \{n \in \mathbb{N} : \omega(n) \geq k\}.$$

That is, W_k consists of all positive integers n whose number of distinct prime factors is larger than or equal to k. In the present paper, we shall continue the research from [7]. In fact, let p_r be the r-th prime number and let $\alpha = p_{r-1}/p_r$. Let F, G be two arithmetic functions formed by γ and

φ. Throughout our discussion, we will mostly deal with the sign changes of $p_r F(n) - p_{r-1} G(n)$, where $n \in W_k$. More precisely, we will prove that there are infinitely many $n \in W_k$ for which,

$$p_r F(n) > p_{r-1} G(n), \tag{2}$$

and also there are infinitely many $m \in W_k$ for which,

$$p_r F(m) < p_{r-1} G(m). \tag{3}$$

Or, equivalently, we will construct two infinite sequences $\{n_i\}_{i\geq 1}, \{m_i\}_{i\geq 1} \subset W_k$ such that,

$$\frac{F(n_i)}{G(n_i)} > \frac{p_{r-1}}{p_r} > \frac{F(m_i)}{G(m_i)}, \text{ for } i = 1, 2, \ldots.$$

Thus, as a continuation of our works [7, 8], we confine the number p_{r-1}/p_r from the right and from the left by an infinity of rational numbers for each. Moreover, in this paper we will study possibilities for proving (2) and (3) infinitely often over some infinite subsets of W_k.

The following section describes some important definitions and theorems that are needed throughout this paper.

2 Basic Tools and Preliminaries

Throughout this work, $2 = p_1 < p_2 < \ldots < p_n < \ldots$ will denote the successive prime numbers. The sequence $q_1 < q_2 < \ldots < q_n < \ldots$ denotes an arbitrary sequence of primes. We also denote by d_{n-1} the gap between p_n and p_{n-1} for $n \geq 2$. That is,

$$d_{n-1} = p_n - p_{n-1}. \tag{4}$$

Note that the Prime Number Theorem (PNT) is equivalent to $p_n \sim n \log n$, and hence,

$$\lim_{n\to+\infty} \frac{p_n}{d_n} = +\infty. \tag{5}$$

The nonstandard version of (5) is given by the following formula,

$$\frac{p_n}{d_n} \simeq +\infty. \tag{6}$$

Then for every $k \geq 1$, we can choose two positive integers r, s with $r \geq 2$ such that,

$$p_k < \left(1 + \frac{p_{r-1}}{d_{r-1}}\right)^{\frac{1}{s}}. \tag{7}$$

For example, assume that $k = 5216954$. It is enough to choose,

$$p_{r-1} = 4944592681451939331717 2147,$$

where

$$p_r = 49445926\,814519\,393317172203.$$

We see that $d_{r-1} = p_r - p_{r-1} = 56$. Thus, for $s = 3$, we obtain,

$$p_k = 89999999 < \left(1 + \frac{p_{r-1}}{d_{r-1}}\right)^{\frac{1}{s}} = \left(1 + \frac{4944592681451939331717214 7}{56}\right)^{\frac{1}{3}},$$
$$= 90000000.59.$$

Next, we state the explicit formula of the Jordan generalization of Euler's function in terms of the standard factorization of n [17, p. 194]. That is, the multiplicative function,

$$\varphi_s(n) = n^s \prod_{p|n} \left(1 - \frac{1}{p^s}\right), \quad \varphi_s(1) = 1, \tag{8}$$

where $\varphi_1 = \varphi$. Moreover, $\varphi_s(n)$ is even for every $n > 2$. Recall that for any $m, n \in \mathbb{N}$, we have,

$$\varphi(mn) = \frac{d}{\varphi(d)} \cdot \varphi(m)\varphi(n), \tag{9}$$

where $d = (m, n)$. We also note that if n is divisible by m, then $\varphi(n)$ is divisible by $\varphi(m)$. Further, we will use the subset $A_{r,s}$ defined for any $r, s \in \mathbb{N}$ by,

$$A_{r,s} = \{n \in \mathbb{N} : p_r \varphi_s(n) > p_{r-1} n^s\}. \tag{10}$$

We recall Dirichlet's Theorem, Bertrand's Theorem and a result on good primes, which will be applied in our proofs.

Theorem 2.1 (Theorem of Dirichlet about primes, [15, p. 347]). *If a and b are relatively prime integers with $a \geq 1$, then the sequence $an + b$ includes infinitely many primes.*

Theorem 2.2 (Theorem of Bertrand [21, p. 24]). *For any $n \in \mathbb{N}$, there exists a prime number $p \in [n, 2n]$.*

Recall that Erdös and Strauss call a prime p_n good if $p_n^2 > p_{n-i} p_{n+i}$ for all values of i from 1 to $n - 1$, see [20, p. 119]. The sequence of good primes starts with $5, 11, 17, 29, \ldots$. In this context, we need to use the following result:

Theorem 2.3 ([9, p. 32]). *There are infinitely many good primes.*

Also, we will use the following notation:

Notation 2.1. *Let n be a positive integer, and let $F : \mathbb{N} \to \mathbb{R}$ be a number-theoretic function.*

- *For every positive integer N, we denote by F^N the arithmetic function given by $F^N(n) = (F(n))^N$.*
- *We write $p^a \parallel n$ if p^a is the largest power of p that divides the integer n, that is, p^a divides n but p^{a+1} does not divide n.*

Now, we explain the nonstandard settings [4, 6] and notations used in this paper. The study of small quantities was first established by Leibniz, the first mathematician to attempt to articulate clearly the concept of small quantities, also known as infinitesimal numbers. Robinson developed the notion of *infinitesimal* in his book [19] on *Nonstandard Analysis*. Another presentation of the nonstandard analysis [12], called **IST** (Internal Set Theory), introduced by Edward Nelson in 1977

[16] which was based on **ZFC** to which is added a new unary predicate called *"standard"*. This predicate gave us the following three axioms: *Transfer principle*, *Idealization principle* and *Standardization principle*.

Recall that any real number used in a classical way is necessarily standard. Thus, 0, $1, \ldots$, 10^{1000}, $\pi, \ldots$, $1/e$ are standard. But not all real numbers are standard.

Definition 2.1 ([5]). *Let x be a real number.*

1. *x is said to be limited if there exists a standard $n \in \mathbb{N}$ such that $|x| \leq n$.*
2. *x is said to be unlimited, or infinitely large, if x is not limited. That is, $|x| > n$ for every standard $n \in \mathbb{N}^*$ which we denote by $x \simeq +\infty$.*
3. *x is said to be infinitesimal, or infinitely small, if $|x| \leq 1/n$ for every standard $n \in \mathbb{N}^*$, in which case we write $x \simeq 0$.*
4. *x is is said to be appreciable if x is limited but not infinitesimal.*

Recall that $\mathbb{N}^\sigma$ denotes the set of all limited positive integers. We denote by ϕ and $\pounds$ for an arbitrary infinitesimal real number and an arbitrary limited real number, respectively. For details, see [5, p. 3]. Clearly, if ω is unlimited, then $1/\omega$ is infinitesimal. The converse is true, namely if ϕ is infinitesimal then, $1/\phi$ is unlimited. Further, we have the following facts:

- $a \in \mathbb{Z}$ is limited if and only if a is standard.
- If x is limited and y is unlimited and positive, then $x < y$.
- If ϕ is infinitesimal and a is appreciable, then $\phi < |a|$. Thus, zero is the only standard infinitesimal real number.

Note that any formula in the language **ZFC** is called *internal*. However, a formula of the nonstandard language **IST** which deals the new predicate standard is called *external*. The formulas: $[x < \varepsilon \Rightarrow x < 2\varepsilon]$, $[0, +\infty) \subset \mathbb{R}$ and $\varepsilon \neq 0$ are internal because the symbols $<, 2, \varepsilon, 0, +\infty, \subset \mathbb{R}, \neq$ are definable in the language **ZFC**. Examples of external formulas are the formulas: $p \in \mathbb{N}$ is prime and $p \simeq +\infty$, $x \simeq +\infty \Rightarrow x^2 \simeq +\infty$, $\phi \simeq 0$ and $\forall a \in \mathbb{R}$ (a is limited $\nRightarrow$ a is appreciable).

Note that external sets are outside **IST**, however, a complimentary axiomatic is given in [12]. We conclude that $\{x \in \mathbb{R}; |x| \leq \varepsilon\}$, $\{p \in \mathbb{N}; p \text{ is prime and } p > 1/\varepsilon\}$ are internal whereas the two sets $\{x \in \mathbb{R}; x \simeq 0\}$ and $\{p \in \mathbb{N}; p \text{ is prime and } p \simeq +\infty\}$ are external. Observe that a set defined by means of an external formula is not necessarily external because sometimes an external formula is equivalent to an internal formula. For example, in [5], it is shown that the set $\{x \in \mathbb{R} : x \text{ is standard and } x \simeq 0\}$ which is equal to 0, is both internal and standard.

We recognize that in nonstandard literature we find several points of view to define an external set [12, 18] and this reflects the problematic posed by this notion. In this article, we opt for the following definition.

The following theorems are important for the proof of our results.

Theorem 2.4 (Cauchy's principle [5, p. 19]). *No external set is internal.*

Theorem 2.5 ([4, p. 16]). *Let $(u_n)_{n\geq 1}$ be a standard sequence of elements of $\mathbb{R}$. Then, $(u_n)_{n\geq 1}$ converges to l if and only if $u_n \simeq l$ for all unlimited n.*

As a consequence of the PNT in the form $p_n \sim n \log n$, Cauchy's principle and Theorem 2.5 is given in the following example.

Example 2.1. *Let n be unlimited. By using the PNT and Theorem 2.5, $p_n/p_{n-1} = 1 + \phi_n$, where ϕ_n is infinitesimal positive. Thus, for every limited prime number q, we have $1 + \frac{1}{q} > 1 + \phi_n$. It follows from Cauchy's principle that there exists an unlimited prime number $\widetilde{q}$ such that $1/\widetilde{q} > \phi_n$. Then, $\widetilde{q} \leq [1/\phi_n] \simeq +\infty$ where the expression $[x]$ represents the integer part of x.*

In another example, we show that if ω is an unlimited positive integer, then there exists an unlimited prime number p such that $p < \omega$.

Example 2.2. *Let ω be an unlimited positive integer, and consider the internal set $L = \{n \in \mathbb{N}\,;\, p_n < \omega\}$. Since the sequence $(p_n)_{n\geq 1}$ of prime numbers is standard, L contains all the standard integers. Then by Cauchy's principle, L contains an unlimited positive integer ν. Hence, $p_\nu < \omega$. Since the sequence $(p_n)_{n\geq 1}$ is increasing, we conclude that p_ν is unlimited.*

Note that the use of positive integers having sufficiently large number of distinct prime factors was made precise in the framework of nonstandard analysis by interpreting "sufficiently" large as unlimited. Let us define two infinite subsets of positive integers that we shall use at the end of Section 3 and Section 4.

Definition 2.2. *Let $\overline{W}_\infty$ be the subset of $\mathbb{N}$ given by,*

$$\overline{W}_\infty = \{n \in \mathbb{N} : \omega(n) \simeq +\infty\}.$$

That is, $\overline{W}_\infty$ has only unlimited positive integers n, where its number of distinct prime factors is also unlimited. Let $W_\infty \subset \overline{W}_\infty$ denotes the set of all unlimited positive integers n satisfying the following two conditions:

1. $\omega(n) \simeq +\infty$,
2. *Any divisor d of n with $d \neq 1$ is unlimited. That is, any proper divisor of n is unlimited.*

Thus, W_∞ has only odd numbers. Also, for any limited $k \geq 1$, we see that,

$$\ldots \subset W_\infty \subset \overline{W}_\infty \subset W_k \subset W_{k-1} \subset \ldots \subset W_1 = \mathbb{N}_0,$$

where $W_k \nsubseteq \overline{W}_\infty$.

Let $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$, where $q_1, q_2, \ldots, q_s$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{N}$ are positive. We have,

- $n \in \overline{W}_\infty$ if and only if $s \simeq +\infty$.
- $n \in W_\infty$ if and only if $s \simeq +\infty$ and $q_i \simeq +\infty$ for $i = 1, 2, \ldots, s$.

It is clear that if a Diophantine inequality has infinitely many solutions on the set W_k, it may not necessarily have infinitely many solutions on the set W_{k+1}, and so on. For example in [7], under the condition (7), it is proved that $p_r\varphi_s(n) - p_{r-1}n^s$ has infinitely many sign changes on the set W_k. But, it does not follow directly that the same expression has infinitely many sign changes on the subsets $\overline{W}_\infty, W_\infty$. These questions and other similar questions are addressed in this work.

This manuscript is organized in the following way: In Section 2, we introduce some basic facts and notations that will appear in the proof of our results. In Section 3, we will prove that $p_r\gamma^N(n)-p_{r-1}\gamma^N(n+l)$ has infinitely many sign changes, where n belongs to a certain infinite subset of W_k. In the proof, we will use Dirichlet's Theorem about primes in arithmetic progressions and Bertrand's Theorem. In the framework of internal set theory, the same expression will be studied using infinite external subsets. Finally, in Section 4, we take a look at the corresponding question for $p_r\varphi_s(n)-p_{r-1}n^s$, where $s\geq 1$.

We are also concerned with the set $A_{r,s}$ given in (10). For example, in Theorem 4.1, we show that there are infinitely many $n\in W_\infty$ such that $n\in A_{r,s}$. In the case when r is unlimited and s is limited, we prove the existence of a multiply perfect number N with $N\notin A_{r,s}$. Moreover, under some conditions, we are able to show that $p_r\varphi_s(n)-p_{r-1}n^s$ changes sign infinitely often on the set W_∞. The method of proof involves some results in elementary number theory and nonstandard analysis as well as the Prime Number Theorem and Cauchy's Principle for showing that if r is unlimited and s is limited, then $p_r\varphi_s(n)<p_{r-1}n^s$ holds for infinitely many $n\in W_\infty$.

3 Sign Changes Using the Kernel of Positive Integers

Let $l,N,r\in\mathbb{N}$ with $r\geq 2$. Throughout this section, for any $n\geq 1$, we study the sign changes for $\gamma^N(n)$ and $\gamma^N(n+l)$. Note that $p_r\gamma^N(n)>p_{r-1}\gamma^N(n+l)$ does not guarantee that we have $p_r\gamma(n)>p_{r-1}\gamma(n+l)$, since $\gamma^N(n)\geq\gamma(n)$ and $\dfrac{p_{r-1}}{p_r}\leq\left(\dfrac{p_{r-1}}{p_r}\right)^{\frac{1}{N}}$. Thus, the fact that $p_r\gamma^N(n)-p_{r-1}\gamma^N(n+l)$ changes sign infinitely often does not follow from the fact that $p_r\gamma(n)-p_{r-1}\gamma(n+l)$ changes sign infinitely often. In addition, for some infinite subsets of positive integers U,V with $U\cap V=\emptyset$, we may derive the existence two infinite increasing sequences (n_i) and (m_i) such that $p_r\gamma^N(n_i)>p_{r-1}\gamma^N(n_i+l)$ and $p_r\gamma^N(m_i)<p_{r-1}\gamma^N(m_i+l)$ for $i\geq 1$.

We first deal with the inequality $p_r\gamma^N(n)>p_{r-1}\gamma^N(n+l)$, where n is square-free.

Proposition 3.1. *Let $l,k,N\geq 1$. There exists an infinite subset W of W_k which has the property: for each $n\in W$, one has,*

$$\begin{cases} p_r\gamma^N(n)>p_{r-1}\gamma^N(n+l), & \text{if } n \text{ is square-free},\\ p_r\gamma^N(n)<p_{r-1}\gamma^N(n+l), & otherwise. \end{cases}$$

For the proof we need the following lemma. First, let C_N^i be the binomial coefficients,

$$C_N^i=\frac{N!}{i!\,(N-i)!}\quad\text{for}\quad 0\leq i\leq N.$$

Lemma 3.1. *Let d_{r-1} be given by (4). Let $(q_1,q_2,\ldots,q_k)$ be a k-tuple of distinct primes such that,*

$$q_1q_2\ldots q_k>l\left(2^N-1\right)\frac{p_{r-1}}{d_{r-1}},\tag{11}$$

and

$$(q_1q_2\ldots q_k,l)=1,\tag{12}$$

where $l,k,N\geq 1$. Then,

$$\sum_{i=1}^{N}\frac{C_N^i l^i}{(q_1q_2\ldots q_k)^i}<\frac{d_{r-1}}{p_{r-1}},$$

for $i = 1, 2, \ldots, N$.

Proof. From Bertrand's Theorem, $\dfrac{p_{r-1}}{d_{r-1}} > 1$. Hence, by (11), we have,

$$(q_1 q_2 \ldots q_k)^i > l^i \left(2^N - 1\right) \frac{p_{r-1}}{d_{r-1}}, \text{for every } i \geq 1,$$

and so,

$$\frac{C_N^i l^i}{(q_1 q_2 \ldots q_k)^i} < \left(\frac{1}{2^N - 1} \frac{d_{r-1}}{p_{r-1}}\right) C_N^i. \tag{13}$$

On the other hand, since $\sum\limits_{i=0}^{N} C_N^i = 2^N$, it follows, from (13) that,

$$\sum_{i=1}^{N} \frac{C_N^i l^i}{(q_1 q_2 \ldots q_k)^i} < \frac{\sum\limits_{i=1}^{N} C_N^i}{2^N - 1} \frac{d_{r-1}}{p_{r-1}} = \frac{d_{r-1}}{p_{r-1}}. \tag{14}$$

As required. □

Proof of Proposition 3.1. Let $W' = \{q_1 q_2 \ldots q_k\}$ be the set which contains one element $q_1 q_2 \ldots q_k$, where $q_1, q_2, \ldots, q_k$ are distinct primes as in (11) and (12), and define,

$$W'' = \{t \in \mathbb{N} \ : \ q_1 q_2 \ldots q_k \, t + l \text{ is prime } \}.$$

It follows from Theorem 2.1 that W'' is infinite. Let $W = W'W''$, where the product set $W'W''$ consists of all elements ab with $a = q_1 q_2 \ldots q_k$ and $b \in W''$. Clearly, W is an infinite subset of W_k.

Now, let $n_t = q_1 q_2 \ldots q_k \, t \in W$, i.e., $t \in W''$. We see that,

$$\frac{\gamma^N (n_t + l)}{\gamma^N (n_t)} = \frac{(n_t + l)^N}{\gamma^N (n_t)} = \left(\frac{n_t}{\gamma (n_t)}\right)^N + \varepsilon_{n_t},$$

where

$$\varepsilon_{n_t} = \frac{\sum\limits_{i=1}^{N} C_N^i n_t^{N-i} l^i}{\gamma^N (n_t)} = \left(\frac{n_t}{\gamma (n_t)}\right)^N \sum_{i=1}^{N} \frac{C_N^i l^i}{n_t^i}.$$

Therefore,

$$\frac{\gamma^N (n_t + l)}{\gamma^N (n_t)} = \begin{cases} 1 + \sum\limits_{i=1}^{N} \dfrac{C_N^i l^i}{n_t^i}, & \text{if } n_t \text{ is square-free,} \\ \prod\limits_{p^a \| n_t a > 1} p^{N(a-1)} + \varepsilon_{n_t}, & \text{otherwise.} \end{cases} \tag{15}$$

From Lemma 3.1 and since $n_t \geq q_1 q_2 \ldots q_k$, we obtain,

$$\sum_{i=1}^{N} \frac{C_N^i l^i}{n_t^i} \leq \sum_{i=1}^{N} \frac{C_N^i l^i}{(q_1 q_2 \ldots q_k)^i} < \frac{d_{r-1}}{p_{r-1}}. \tag{16}$$

In the case when n_t is square-free, by (15) and (16), we have,

$$\frac{\gamma^N(n_t+l)}{\gamma^N(n_t)} = 1+\sum_{i=1}^{N}\frac{C_N^i l^i}{n_t^i} \leq 1+\sum_{i=1}^{N}\frac{C_N^i l^i}{(q_1q_2\dots q_k)^i} < \frac{p_r}{p_{r-1}},$$

and in the case when n_t is not square-free, by (15) and Bertrand's Theorem we also have,

$$\frac{\gamma^N(n_t+l)}{\gamma^N(n_t)} = \prod_{\substack{p^a\|n_t\\ a>1}} p^{N(a-1)} + \varepsilon_{n_t} \geq 2+\varepsilon_{n_t} > \frac{p_r}{p_{r-1}}.$$

The proof is finished. □

Remark 3.1. *In view of the proof of Proposition 3.1, we can not deduce that $p_r\gamma^N(n)-p_{r-1}\,\gamma^N(n+l)$ changes sign infinitely often on the set $W=W'W''$. Of course, at least one of the inequalities $p_r\gamma^N(n) > p_{r-1}\,\gamma^N(n+l)$ and $p_r\gamma^N(n) < p_{r-1}\,\gamma^N(n+l)$ holds infinitely often. In fact, it is an open question as to whether there are infinitely many primes of the form $q_1q_2\dots q_k\,t+l$ such that t is square-free. Also, it is not known whether there are infinitely many primes of the same form such that t is not square-free.*

Using a similar reasoning as in Proposition 3.1 and the fact that the set of all k-tuples $(q_1,q_2,\dots,q_k)$ satisfying (11) is infinite, we can prove the following result:

Theorem 3.1. *Let p_r be the r-th prime number with $r\geq 2$ and let $k,l;N\geq 1$. Then, $p_r\gamma^N(n)-p_{r-1}\,\gamma^N(n+l)$ has infinitely many sign changes on the set W_k.*

Proof. We prove that each of the inequalities,

$$p_r\gamma^N(n) > p_{r-1}\,\gamma^N(n+l) \quad \text{and} \quad p_r\gamma^N(n) < p_{r-1}\,\gamma^N(n+l),$$

holds for infinitely many $n\in W_k$. We show the first inequality by using the fact that the set of k-tuples satisfying (11) is infinite. Let $(q_1,q_2,\dots,q_k)$ be a k-tuple of distinct primes satisfying (11). We put $q_k=p_{i_0}$, where p_{i_0} is the i_0-th prime number, and define n_s for any $s\geq 0$ by,

$$n_s = q_1q_2\dots q_kp_{i_0+1}\dots p_{i_0+s}, \tag{17}$$

which is square-free. Moreover, we see that $n_s\in W_k$. Since,

$$\gamma(n_s+l)\leq n_s+l \quad \text{and} \quad \gamma(n_s)=n_s\geq q_1q_2\dots q_k,$$

we have by Lemma 3.1,

$$\begin{aligned}\frac{\gamma^N(n_s+l)}{\gamma^N(n_s)} &\leq \frac{(n_s+l)^N}{n_s^N} \qquad (18)\\ &= 1+\frac{\sum_{i=1}^{N}C_N^i n_s^{N-i}l^i}{n_s^N}\\ &= 1+\sum_{i=1}^{N}\frac{C_N^i l^i}{n_s^i}\\ &\leq 1+\sum_{i=1}^{N}\frac{C_N^i l^i}{(q_1q_2\dots q_k)^i} < \frac{p_r}{p_{r-1}} = 1+\frac{d_{r-1}}{p_{r-1}}. \qquad (19)\end{aligned}$$

Note that the prime numbers $q_1, q_2, \ldots, q_k$ satisfy (11), and so the right-hand side of (19) holds. Combining (18) and (19), we obtain,

$$\frac{\gamma^N(n_s + l)}{\gamma^N(n_s)} < \frac{p_r}{p_{r-1}}.$$

The desired inequality is proved.

We next show the second inequality. Let $(q_1, q_2, \ldots, q_k)$ be a k-tuple of distinct primes satisfying (12). By Theorem 2.1, there are infinitely many primes of the form $q_1^2 q_2^2 \ldots q_k^2\, t + l$. Let p be a prime of this form, and set $n_t = p - l = q_1^2 q_2^2 \ldots q_k^2\, t$ which is an element of W_k (i.e., there are infinitely many such numbers). Therefore,

$$\begin{aligned}\frac{\gamma^N(n_t + l)}{\gamma^N(n_t)} &= \frac{(n_t + l)^N}{\gamma^N(n_t)} \qquad (20)\\ &= \frac{n_t^N}{\gamma^N(n_t)} + \sum_{i=1}^{N} \frac{C_N^i n_t^{N-i} l^i}{\gamma^N(n_t)}\\ &= \frac{n_t^N}{\gamma^N(n_t)} \left(1 + \frac{\sum_{i=1}^{N} C_N^i l^i}{n_t^i}\right)\\ &> \frac{n_t^N}{\gamma^N(n_t)} \geq \frac{n_t}{\gamma(n_t)}. \qquad (21)\end{aligned}$$

Now, it suffices to prove that,

$$\frac{n_t}{\gamma(n_t)} \geq q_1 q_2 \ldots q_k \frac{t}{\gamma(t)}.$$

Suppose we have $t = a.b$ such that,

$$a = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k};\ \alpha_i \geq 0\,,\ \text{for } i = 1, 2, \ldots, k,$$

and

$$b = l_1^{\beta_1} l_2^{\beta_2} \ldots l_s^{\beta_s};\ \beta_j \geq 0\,,\ \text{for } j = 1, 2, \ldots, s,$$

where $l_1, l_2, \ldots, l_s$ are distinct primes satisfying $l_j \notin \{q_1, q_2, \ldots, q_k\}$, for $j = 1, 2, \ldots, s$. Since $\gamma(b) \leq \gamma(t) \leq t$ and $(a, b) = 1$, it follows from Bertrand's Theorem that,

$$\frac{n_t}{\gamma(n_t)} = \frac{q_1^2 q_2^2 \ldots q_k^2\, t}{\gamma(q_1^2 q_2^2 \ldots q_k^2\, t)} = q_1 q_2 \ldots q_k \frac{t}{\gamma(b)} \geq q_1 q_2 \ldots q_k \frac{t}{\gamma(t)} \geq 2 > \frac{p_r}{p_{r-1}}. \qquad (22)$$

Combining (20), (21) and (22), we get,

$$\frac{\gamma^N(n_t + l)}{\gamma^N(n_t)} > \frac{p_r}{p_{r-1}}.$$

Thus, the desired inequality holds infinitely often. This completes the proof. □

Now, we prove the existence of an infinite proper subset of W_k for which $p_r \gamma^N(n) - p_{r-1}\, \gamma^N(n + l)$ changes sign infinitely often. Let $A_l(c)$ and $B_l(c)$ be the subsets of $\mathbb{N}$ given by,

$$A_l(c) = \left\{n \in \mathbb{N} \,:\, 2^{c-1} \mid n + l, 2^c \nmid n \text{ and } 2^c \nmid n + l\right\},$$

and $B_l(c) = \left\{n \in \mathbb{N} \,:\, 2^{c-1} \mid n, 2^c \nmid n \text{ and } 2^c \nmid n + l\right\}$.

Theorem 3.2. *Let c, k, l, N, r be positive integers with $c, r \geq 2$. If $k \geq 3$ and l is odd, then,*

1. *$W_k \cap A_l(c), W_k \cap B_l(c)$ are infinite.*
2. *$p_r \gamma^N(n) - p_{r-1}\gamma^N(n+l)$ has infinitely many sign changes on the set $W_k \cap (A_l(c) \cup B_l(c))$.*

Proof. We prove the theorem as follows:

1. We show that $W_k \cap A_l(c)$ is infinite. Let s be integer with $s \geq k$. Since,
$$(2^c, 1) = 1, \quad \left(2^c, 2^{c-1}+1\right) = 1 \quad \text{and} \quad (2^c, l) = 1,$$
by Theorem 2.1, there exists an s-tuple of distinct primes $(q_1, q_2, ..., q_s)$ such that $q_1, q_2, \ldots, q_{s-2}$ are all of the form $2^c t + 1$, q_{s-1} is of the form $2^c t + 2^{c-1} + 1$ and q_s is of the form $2^c t - l$, respectively. We set,
$$\begin{cases} q_1 q_2 \ldots q_{s-2} = 2^{c-1}t + 1, \\ q_{s-1} = 2^{c-1}t' + 1, \\ q_s = 2^{c-1}t'' - l, \end{cases}$$
for some positive integers t, t', t'' with t, t'' are even and t' is odd. It follows that,
$$\begin{aligned} q_1 q_2 \ldots q_{s-2} q_{s-1} q_s &= \left(2^{c-1}t + 1\right)\left(2^{c-1}t' + 1\right)\left(2^{c-1}t'' - l\right) \\ &= 2^{c-1}\left(t'' + 2^{c-1}tt'' + 2^{c-1}t't'' + 2^{c-2}tt't'' - 2^{c-1}ltt' - tl - t'l\right) - l \\ &= 2^{c-1}b - l, \end{aligned}$$
where $b = t''\left(1 + 2^{c-1}t + 2^{c-1}t' + 2^{c-2}tt'\right) - l\left(t + t' + 2^{c-1}tt'\right)$ is odd. Define the square-free numbers,
$$n_s = q_1 q_2 \ldots q_s = 2^{c-1}b - l, \tag{23}$$
for some odd $b \geq 1$ with $s \geq k$. We see that there are infinitely many s such that $n_s \in W_k$, 2^{c-1} divides $n_s + l$ and 2^c does not divide neither n_s nor $n_s + l$.

2. We show that $W_k \cap B_l(c)$ is infinite. Let $(2, q_2, \ldots, q_k)$ be a k-tuple of distinct primes such that $(2q_2 \ldots q_k, l) = 1$. We can easily prove that,
$$\left(2^c q_2 \ldots q_k, 2^{c-1}q_2 \ldots q_k + l\right) = 1. \tag{24}$$
In fact, assume that $\left(2^c q_2 \ldots q_k, 2^{c-1}q_2 \ldots q_k + l\right) = d > 1$. Note that d can not be even because d divides $2^{c-1}q_2 \ldots q_k + l$ which is odd. Hence, d is divisible by at least one prime q_{i_o} for some $i_0 \in \{2, \ldots, k\}$, and therefore,
$$q_{i_o} \mid 2^{c-1}q_2 \ldots q_k + l.$$
That is, q_{i_o} divides l which is impossible since $(q_2 q_3 \ldots q_k, l) = 1$. This proves (24). Thus, there exist infinitely many primes of the form,
$$2^c q_2 \ldots q_k t + 2^{c-1}q_2 \ldots q_k + l.$$
On the other hand, we see that,
$$2^c q_2 \ldots q_k t + 2^{c-1}q_2 \ldots q_k + l = 2^{c-1}q_2 \ldots q_k (2t+1) + l. \tag{25}$$
This means that there exist infinitely many primes of the form $2^{c-1}q_2 \ldots q_k t + l$, where t is odd. For each such integer t, let $n_t = 2^{c-1}q_2 \ldots q_k t$. Hence, $n_t \in W_k$, 2^{c-1} divides n_t and 2^c does not divide neither n_t nor $n_t + l$.

3. We prove the inequality $p_r\gamma^N(n) > p_{r-1}\gamma^N(n+l)$ for infinitely many $n \in W_k \cap A_l(c)$. In fact, let $(q_1, q_2, \ldots, q_s)$ be a s-tuple of distinct primes satisfying both (11) and (23). That is, $q_1q_2\ldots q_s$ is strictly larger than $l\left(2^N-1\right)\dfrac{p_{r-1}}{d_{r-1}}$ and $q_1q_2\ldots q_s$ is of the form $2^{c-1}b-l$, where b is odd and $s \geq k$. Taking $n_s = q_1q_2\ldots q_s$ which is an element of $W_k \cap A_l(c)$, the result we must prove follows immediately by applying the inequalities stated in the first part of the proof of Theorem 3.1. Indeed, n_s is square-free as in (17), and so the inequality $p_r\gamma^N(n_s) > p_{r-1}\gamma^N(n_s+l)$ comes from (18) and (19).

4. We prove the inequality $p_r\gamma^N(n) < p_{r-1}\gamma^N(n+l)$ for infinitely many $n \in W_k \cap B_l(c)$. Let $(2, q_2, \ldots, q_k)$ be a k-tuple of distinct primes satisfying (12). By the same method used in the proof of (24), we have,

$$\left(2^c q_2^2\ldots q_k^2, 2^{c-1}q_2^2\ldots q_k^2 + l\right) = 1,$$

from which it follows that there are infinitely many primes of the form,

$$2^{c-1}q_2^2\ldots q_k^2 t + l,$$

where t is odd. We let $n_t = 2^{c-1}q_2^2\ldots q_k^2 t$ which is an element of $W_k \cap A_l(c)$. By applying (20), (21) and (22) for $n_t = 2^{c-1}q_2^2\ldots q_k^2 t$ we obtain the inequality $p_r\gamma^N(n_t) < p_{r-1}\gamma^N(n_t+l)$.

Hence, by the above, $p_r\gamma^N(n+l) - p_{r-1}\,\gamma^N(n)$ has infinitely many sign changes on the set $(W_k \cap A_l(c)) \cup (W_k \cap B_l(c))$. The proof of Theorem 3.2 is now complete. □

Next, we deal with the sign changes of $p_r\gamma^N(n+l) - p_{r-1}\,\gamma^N(n)$ using some infinite external subsets of W_k. We will start by proving the following proposition.

Theorem 3.3. *The sets $\overline{W}_\infty$ and W_∞ are external.*

Proof. Suppose, by way of contradiction, that $\overline{W}_\infty$ is internal. Since $p_1p_2\ldots p_n \notin \overline{W}_\infty$ for every $n \in \mathbb{N}^\sigma$ $(n \geq 1)$, by Cauchy's principle, there exists an unlimited positive integer ω such that $p_1p_2\ldots p_\omega \notin \overline{W}_\infty$. This is a contradiction since $p_1p_2\ldots p_\omega$ having ω prime factors, and by Definition 2.2, $p_1p_2\ldots p_\omega \in \overline{W}_\infty$.

Similar argument shows that W_∞ is external. Suppose, to the contrary, that W_∞ is internal. Let ω be an unlimited integer and let p_ω be the ω-th prime number which is necessarily unlimited. Consider the set,

$$\{n \in \mathbb{N} \,:\, p_\omega p_{\omega+1}\ldots p_{\omega+n} \notin W_\infty\},$$

which is internal and contains all the standard integers of $\mathbb{N}$. By Cauchy's principle, this set contains an unlimited positive integer ν, that is, $p_\omega p_{\omega+1}\ldots p_{\omega+\nu} \notin W_\infty$. This is a contradiction since $p_\omega p_{\omega+1}\ldots p_{\omega+\nu}$ clearly belongs to W_∞. □

Let us use the external subsets $\overline{W}_\infty$ and W_∞ to obtain nonclassical results.

Theorem 3.4. *Let c, k, l, r be positive integers with $c \geq 3$ and $r \geq 2$. If $k \geq 3$ and l is odd, then $p_r\gamma(n+l) - p_{r-1}\,\gamma(n)$ has infinitely many sign changes on the set $\overline{W}_\infty \cap (A_l(c) \cup B_l(c))$.*

We need the following lemma.

Lemma 3.2. *Let $r \geq 2$. For every $\phi \simeq 0$,*

$$\frac{p_r}{p_{r-1}} = 1 + \frac{d_{r-1}}{p_{r-1}} < 2 - \phi. \tag{26}$$

Proof. If r is unlimited, then by (6) $\dfrac{d_{r-1}}{p_{r-1}} \simeq 0$ from which it follows that $1 + \dfrac{d_{r-1}}{p_{r-1}} < 2 - \phi$ for every $\phi \simeq 0$. But, if r is limited, then $\dfrac{d_{r-1}}{p_{r-1}}$ is standard and strictly less than 1 from Bertrand's Theorem. In this case, $\dfrac{d_{r-1}}{p_{r-1}} < 1 - \phi$ for every $\phi \simeq 0$; otherwise, $\dfrac{d_{r-1}}{p_{r-1}} \geq 1 - \phi'$ for some $\phi' \simeq 0$, from which we get $2 - p_r/p_{r-1} \leq \phi'$, i.e., $p_r/p_{r-1} \simeq 2$, a contradiction. This proves (26). □

Proof of Theorem 3.4. Let s be an unlimited positive integer. As in the proof of Theorem 3.2, we can show that $p_r\gamma(n+l) - p_{r-1}\gamma(n)$ is positive and negative for infinitely many $n \in W_s \cap (A_l(c) \cup B_l(c))$, which is a subset of $\overline{W}_\infty \cap (A_l(c) \cup B_l(c))$.

First, let $(2, q_2, q_3, \ldots, q_s)$ be a s-tuple of distinct primes satisfying (23). Set,

$$q_1 q_2 \ldots q_s + l = 2^{c-1} N_s \; ; \; (2, N_s) = 1. \tag{27}$$

For $n_s = q_1 q_2 \ldots q_s$, we have $n_s \in W_s \cap A_l(c)$. Moreover, we see that,

$$\begin{aligned} \frac{\gamma(n_s)}{\gamma(n_s + l)} &= \frac{q_1 q_2 \ldots q_s}{\gamma(q_1 q_2 \ldots q_s + l)} = \frac{2^{c-1} N_s - l}{2\gamma(N_s)} \\ &= \begin{cases} 2^{c-2} - \dfrac{l}{2\gamma(N_s)}, & \text{if } N_s = \prod\limits_{p \| N_s} p, \\ 2^{c-2} \prod\limits_{\substack{p^a \| N_s \\ a>1}} p^{a-1} - \dfrac{l}{2\gamma(N_s)}, & \text{otherwise.} \end{cases} \end{aligned} \tag{28}$$

We distinguish two cases:

Case 1: Assume that $\gamma(N_s)$ is unlimited. Then,

$$\phi_s \equiv \frac{l}{2\gamma(N_s)} \simeq 0.$$

If N_s is square-free, then by Lemma 3.2 we have,

$$\frac{\gamma(n_s)}{\gamma(n_s + l)} = 2^{c-2} - \phi_s \geq 2 - \phi_s > \frac{p_r}{p_{r-1}},$$

and if N_s is not square-free we also have,

$$\frac{\gamma(n_s)}{\gamma(n_s + l)} = 2^{c-2} \prod_{\substack{p^a \| N_s \\ a>1}} p^{a-1} - \phi_s \; \geq 2 - \phi_s > \frac{p_r}{p_{r-1}}.$$

Case 2: Assume that $\gamma(N_s)$ is limited. From (27), it is easy to see that $2^{c-1}N_s$ is unlimited, and so (28) implies,

$$\frac{\gamma(n_s)}{\gamma(n_s + l)} = \frac{2^{c-1} N_s - l}{2\gamma(N_s)} \simeq +\infty,$$

since l is limited. Therefore,

$$\frac{\gamma(n_s)}{\gamma(n_s+l)} > \frac{p_r}{p_{r-1}},$$

the later being limited. Thus, we obtain the desired inequality.

Second, as in (24) and (25), we can choose a s-tuple of distinct primes $(2, q_2, q_3, \ldots, q_s)$ such that the following arithmetic progression:

$$2^{c-1}q_2q_3\ldots q_s\, t + l;\ t \in \mathbb{N},$$

contains infinitely many primes with t is odd. For each such t set $n_t = 2^{c-1}q_2q_3\ldots q_s\, t$. Then $n_t \in W_s$. Moreover, it is clear that $n_t \in B_l(c)$ since $q_2q_3\ldots q_s\, t$ is odd and $n_t + l$ is prime. Thus,

$$\frac{\gamma(n_t)}{\gamma(n_t+l)} \leq \frac{n_t}{n_t+l} < \frac{p_r}{p_{r-1}}. \tag{29}$$

This proves the inequality $p_r\gamma(n_t+l) > p_{r-1}\gamma(n_t)$.

Theorem 3.4 is proved. □

We close this section by proving the following proposition.

Proposition 3.2. *Let $l, N, r \in \mathbb{N}$ with $r \geq 2$. If N, r are limited, then $p_{r-1}\gamma^N(n+l) - d_{r-1}\gamma^N(n)$ changes sign infinitely often on the set $\overline{W}_\infty$.*

Before proving Proposition 3.2, we need the following lemma.

Lemma 3.3. *Let l, m, k be positive integers and let p be an odd prime number. If l is odd, then 2^m divides $n+l$ for infinitely many $n \in W_k$ and if l is even, then p^m divides $n+l$ for infinitely many $n \in W_k$.*

Proof. First, assume that l is odd. For every positive integer $s \geq 2$, there exists a s-tuple of distinct primes $(q_1, q_2, \ldots q_s)$ such that q_i $(1 \leq i \leq s-1)$ is of the form $2^m t+1$ and q_s is of the form $2^m t-l$. Therefore, the integer $n_s = q_1q_2\ldots q_s$ is of the form $2^m t-l$. This means that $n \in W_k$ and $2^m \mid n+l$. If l is even, then we repeat the same idea to show that there are infinitely many $n \in W_k$ such that p^m divides $n+l$ by taking primes of the form $p^m t+1$ and $p^m t-l$, respectively. □

Proof of Proposition 3.2. It suffices to prove the following statements:

I) There are infinitely many $n \in \overline{W}_\infty$ such that $p_{r-1}\gamma^N(n+l) < d_{r-1}\gamma^N(n)$. Let s be an unlimited positive integer, and choose an unlimited positive integer m for which $\dfrac{2^m}{l} \simeq +\infty$. We distinguish two cases:

Case 1: Assume that l is odd. By Lemma 3.3, there exist primes $(q_i)_{1\leq i\leq s}$ such that 2^m divides $q_1q_2\ldots q_s + l$. We set $q_1q_2\ldots q_s + l = 2^a N_s$, where $(2, N_s) = 1$ and $a \geq m$. For $n_s = q_1q_2\ldots q_s \in \overline{W}_\infty$, we have,

$$\frac{\gamma(n_s)}{\gamma(n_s+l)} = \frac{2^a N_s - l}{2\gamma(N_s)}. \tag{30}$$

Since $\gamma(N_s) \leq N_s$ and $\dfrac{l}{2^a} \simeq 0$, it follows from (30) that,

$$\frac{\gamma^N(n_s)}{\gamma^N(n_s+l)} \geq 2^{a-1}\left(\frac{N_s - \dfrac{l}{2^a}}{N_s}\right)^N = 2^{a-1}(1-\phi_1)^N \simeq +\infty, \tag{31}$$

where $\phi_1 = \dfrac{l}{2^a N_s} \simeq 0$ and $(1-\phi_1)^N \simeq 1$. It follows that,

$$\frac{\gamma^N(n_s)}{\gamma^N(n_s+l)} > \frac{p_{r-1}}{d_{r-1}}, \tag{32}$$

since $\dfrac{p_{r-1}}{d_{r-1}}$ is appreciable.

Case 2: Assume that l is even. By Lemma 3.3, there exist primes $(q_i')_{1\leq i\leq s}$ such that 3^m divides $q_1'q_2'\ldots q_s'+l$. Similarly, assume that $q_1'q_2'\ldots q_s'+l = 3^b N_s'$ where $(3, N_s') = 1$ and $b \geq m$. For $n_s' = q_1'q_2'\ldots q_s'$, using (31) and (32) as above,

$$\frac{\gamma^N(n_s')}{\gamma^N(n_s'+l)} \geq 3^{b-1}\left(\frac{N_s' - \dfrac{l}{3^b}}{N_s'}\right)^N = 3^{b-1}(1-\phi_2)^N > \frac{p_{r-1}}{d_{r-1}},$$

where $\phi_2 = \dfrac{l}{3^b N_s'} \simeq 0$ and $(1-\phi_2)^N \simeq 1$.

II) There are infinitely many $n \in \overline{W}_\infty$ such that $p_{r-1}\gamma^N(n+l) > d_{r-1}\gamma^N(n)$. Indeed, let s be unlimited and let $n_t = q_1q_2\ldots q_s t$ be defined as in the proof of Proposition 3.1, i.e., $n_t + l$ is prime. Therefore,

$$\frac{\gamma^N(n_t)}{\gamma^N(n_t+l)} \leq \left(\frac{n_t}{n_t+l}\right)^N \leq \frac{n_t}{n_t+l} < \frac{p_{r-1}}{d_{r-1}}.$$

Then, clearly the desired inequality holds for infinitely many $n \in \overline{W}_\infty$.

This completes proof of Proposition 3.2. □

4 Sign Changes Using the Generalized Euler's Function

Let $r, s \in \mathbb{N}$ with $r \geq 2$. Throughout this section, we put $F(n) = \varphi_s(n)$ and $G(n) = n^s$ for $n \geq 1$, where φ_s is the Jordan generalization of Euler's function. We will study the sign changes of $p_r\varphi_s(n) - p_{r-1}n^s$, where $n \in W_\infty$. At the end of this section, consider the difference $\varphi_1(n) - 2^t p$, where $t \geq 1$ and p is prime.

Theorem 4.1. *Let p_r and s be as above. There exist infinitely many $n \in W_\infty$ such that $p_r\varphi_s(n) > p_{r-1}n^s$.*

Proof. First, we put $\dfrac{p_{r-1}+1}{p_{r-1}} = 1+\delta_r$, where $\delta_r = \dfrac{1}{p_{r-1}}$. We prove that there exists an unlimited prime number p_u for which,

$$\frac{p_u}{p_u - 1} = 1 + \phi, \tag{33}$$

where for some unlimited positive integer ω it holds that $\phi \leq \dfrac{\delta_r}{\omega} \simeq 0$. In fact, in the case where p_r is limited, we observe that p_u can be any unlimited prime number. Hence, $\dfrac{p_u}{p_u-1} = 1+\phi$ with $\phi \simeq 0$, and so $\omega = \dfrac{1}{\sqrt{\phi}} \simeq +\infty$. In the case where p_r is unlimited, i.e., δ_r is infinitesimal. Here, we let the prime number p_u large enough (for example, $p_u > p_{r-1}^2$) such that $\dfrac{p_u}{p_u-1} = 1 + \dfrac{1}{p_u-1} = 1+\phi$, where $\phi = \dfrac{a}{p_u}$ with $a \in \mathbb{Q}_+^*$ is appreciable. Thus, $\phi < \dfrac{a}{p_{r-1}^2}$, and hence, $\dfrac{p_{r-1}}{a}\phi < \delta_r$ from which it is immediate that $\omega = \dfrac{p_{r-1}}{a} \simeq +\infty$. This proves (33).

Second, let δ_r be as above and define the following internal set,

$$X_r = \{m \in \mathbb{N} \ : \ (1+\phi)^m < 1+\delta_r\},$$

where ϕ satisfies (33). We must prove that X_r contains an unlimited positive integer m_0. We first show that $\mathbb{N}^\sigma$ is a subset of X_r. Indeed, there are only two possibilities to categorize:

Case 1: When p_r is limited, that is, δ_r is standard. For every limited positive integer m, $(1+\phi)^m = 1+\varepsilon$ for some infinitesimal positive ε. Thus, it is clear that $m \in X_r$ since $\varepsilon < \delta_r$.

Case 2: When p_r is unlimited, that is, δ_r is infinitesimal. For every limited positive integer m, we see that $\sum_{i=2}^{m} C_m^i \phi^{i-1} \simeq 0$ because m is limited and ϕ is infinitesimal. On the other hand, we get,

$$\begin{aligned}
(1+\phi)^m &= 1 + \sum_{i=1}^{m} C_m^i \phi^i \\
&= 1 + m\phi + \sum_{i=2}^{m} C_m^i \phi^i \\
&= 1 + \left(m + \sum_{i=2}^{m} C_m^i \phi^{i-1} \right) \phi \\
&< 1 + \omega\phi \\
&\leq 1 + \delta_r.
\end{aligned}$$

This proves that $\mathbb{N}^\sigma$ is a subset of X_r. Therefore, by Cauchy's principle there exists an unlimited positive integer $m_0$such that $m_0 \in X_r$.

Finally, it suffices to prove that there exists a positive integer $n_0 \in W_\infty$ such that $p_r\varphi_s(n_0) > p_{r-1}n_0^s$. In fact, if such integer n_0 exists, then by (8),

$$\frac{\varphi_s\left(n_0^i\right)}{\left(n_0^i\right)^s} = \frac{\varphi_s(n_0)}{n_0^s} > \frac{p_{r-1}}{p_r},$$

and so $p_r\varphi_s\left(n_0^i\right) > p_{r-1}\left(n_0^i\right)^s$ for every $i \geq 1$. Assume by way of contradiction, that for all $n \in W_\infty$, we have $p_r\varphi_s(n) \leq p_{r-1}n^s$. In particular, for $n = p_u p_{u+1} \ldots p_{u+m_0-1} \in W_\infty$

we get,

$$\begin{aligned}\frac{p_r}{p_{r-1}} \leq \frac{n^s}{\varphi_s(n)} &= \frac{n^s}{n^s \prod\limits_{p|n}\left(1-\frac{1}{p^s}\right)} \\ &= \prod_{p|n}\frac{p^s}{p^s-1} \leq \prod_{p|n}\frac{p}{p-1} \\ &\leq \left(\min_{p|n}\left(\frac{p}{p-1}\right)\right)^{\omega(n)} = \left(\frac{p_u}{p_u-1}\right)^{m_0} \\ &= (1+\phi)^{m_0} < \frac{p_{r-1}+1}{p_{r-1}},\end{aligned}$$

where the last inequality comes from the fact that $m_0 \in X_r$. Hence, $p_r < p_{r-1}+1$, which is a contradiction.

This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let $r, s \in \mathbb{N}$ with r is unlimited and s is limited. Then, $p_r\varphi_s(n) < p_{r-1}n^s$ holds for infinitely many $n \in W_\infty$.*

To prove this theorem, we will make use of the following lemma.

Lemma 4.1. *Let $r, s \in \mathbb{N}$ with $r \geq 2$. If $n \notin A_{r,s}$, then $mn \notin A_{r,s}$ for every $m \geq 1$.*

Proof. Let m, n be two positive integers such that $mn \in A_{r,s}$. Then,

$$\frac{p_{r-1}}{p_r} < \frac{\varphi_s(mn)}{(mn)^s} = \prod_{p|mn}\frac{p^s-1}{p^s} \leq \prod_{p|n}\frac{p^s-1}{p^s} = \frac{\varphi_s(n)}{n^s},$$

and therefore $n \in A_{r,s}$. □

Proof of Theorem 4.2. Since $\frac{p_r}{d_{r-1}}$ is unlimited and s is limited, for every limited prime number p we see that,

$$\left(\frac{p_r}{d_{r-1}}\right)^{\frac{1}{s}} \geq p. \tag{34}$$

It follows from Cauchy's principle that (34) holds for some unlimited prime p_m. That is,

$$\left(\frac{p_r}{d_{r-1}}\right)^{\frac{1}{s}} \geq p_m,$$

and so,

$$\frac{p_r - p_{r-1}}{p_r} \leq \frac{1}{p_m^s}.$$

Or, equivalently, $\frac{p_r}{p_{r-1}} \leq \frac{p_m^s}{p_m^s-1}$, which gives $p_r(p_m^s-1) \leq p_{r-1}p_m^s$, i.e., $p_r\varphi_s(p_m) \leq p_{r-1}p_m^s$. Hence, $p_m \notin A_{r,s}$ by (10).

Now, let $\omega \in W_\infty$ be unlimited. By Lemma 4.1, $\omega p_m \notin A_{r,s}$. That is, $p_r \varphi_s(\omega p_m) \leq p_{r-1}(\omega p_m)^s$. On the other hand, since p_r, $p_{r-1}\omega p_m$ are odd and $\varphi_s(\omega p_m)$ is even, we conclude that $p_r \varphi_s(\omega p_m) \neq p_{r-1}(\omega p_m)^s$. Thus, $p_r \varphi_s(\omega p_m) < p_{r-1}(\omega p_m)^s$. This proves that there are infinitely many $n \in W_\infty$ for which $p_r \varphi_s(n) < p_{r-1} n^s$ and so the result. □

Corollary 4.1. *Let $s \geq 1$. There are infinitely many $r \geq 2$ for which $A_{r-1,s}$ contains infinitely many $n \in W_\infty$.*

Proof. First of all, we prove that there are infinitely many $r \geq 2$ such that $A_{r,s} \subset A_{r-1,s}$. In fact, by Theorem 2.3, there are infinitely many good primes. Let p_{r-1} be a prime of this form, where $r \geq 4$. That is, $p_{r-1}^2 > p_{r-2} p_r$. If $n \in A_{r,s}$, then,

$$\frac{n^s}{\varphi_s(n)} < \frac{p_r}{p_{r-1}} < \frac{p_{r-1}}{p_{r-2}}.$$

Therefore, $p_{r-1}\, \varphi_s(n) > p_{r-2} n^s$, i.e., $n \in A_{r-1,s}$ and so $A_{r,s} \subset A_{r-1,s}$. Moreover, by Theorem 4.1, there are infinitely many $n \in W_\infty$ such that $n \in A_{r,s}$ since $A_{r,s}$ has at least an element $n_0 \in W_\infty$ and by (8) we have $n_0^i \in A_{r,s}$ for every $i \geq 1$. Thus, there are infinitely many $n \in W_\infty$ such that $n \in A_{r-1,s}$. This completes the proof. □

As before, for a positive integer n we put $\sigma(n)$ for the sum of its divisors.

Definition 4.1. *A number n is said to be multiply perfect if $\sigma(n) = kn$ for some positive integer k. Here, n is also called a k-perfect number.*

Theorem 4.3 ([20, p. 173])**.** *If p is prime, n is p-perfect and p does not divide n, then pn is $(p+1)$-perfect.*

Note that it is possible to create higher-order perfect numbers from lower-order, based on Theorem 4.3. A simple search with Maple, we see that $\sigma(459818240) = 3 \cdot 459818240$. Thus, 459818240 is 3-perfect which is not divisible by 3, and by the same theorem, $3 \cdot 459818240 = 1379454720$ is 4-perfect, which is true since $\sigma(1379454\,720) = 4 \cdot 1379454720$.

We have the following result.

Theorem 4.4. *Let $r, s \in \mathbb{N}$ with r is unlimited and s is limited. There exists a multiply perfect number N for which $p_r \varphi_s(N) \leq p_{r-1} N^s$. That is, such multiply perfect $N \notin A_{r,s}$.*

Proof. Let p be a limited prime number and let N be a p-perfect number. We distinguish two cases:

Case 1: If p does not divide N, then from Theorem 4.3, the number pN is $(p+1)$-perfect. On the other hand, from the Prime Number Theorem we see that,

$$\left(1 + \frac{p_{r-1}}{d_{r-1}}\right)^{\frac{1}{s}} \simeq +\infty,$$

because r is unlimited and s is limited, from which it is immediate that,

$$\left(1 + \frac{p_{r-1}}{d_{r-1}}\right)^{\frac{1}{s}} > p. \tag{35}$$

This means that $p_r(p^s - 1) < p_{r-1} \cdot p^s$, and so $p \notin A_{r,s}$. From Lemma 4.1, $pN \notin A_{r,s}$.

Case 2: If p divides N, then there exists a positive integer a such that $N = pa$. Since $p \notin A_{r,s}$, it follows from Lemma 4.1 that $pa = N \notin A_{r,s}$.

Thus, in both cases, there is a p-perfect number or a $(p+1)$-perfect number N such that $N \notin A_{r,s}$. This completes the proof of Theorem 4.4. □

Corollary 4.2. *If r is unlimited and s is limited, then $p_r\varphi_s(n) - p_{r-1}n^s$ changes sign for infinitely many $n \in W_\infty$.*

Proof. By Theorems 4.1, the inequality $p_r\varphi_s(n) > p_{r-1}n^s$ holds for infinitely many $n \in W_\infty$ whenever $r \geq 2$ and $s \geq 1$. On the other hand, by Theorem 4.2 the inequality $p_r\varphi_s(n) < p_{r-1}n^s$ holds for infinitely many $n \in W_\infty$ whenever r is unlimited and s is limited. □

Let P be the set of all primes, and let l be a positive integer. We have the following result:

Proposition 4.1. *Let $r \in \mathbb{N}$ with $r \geq 2$. Let A be an infinite external subset of positive integers such that $W_\infty \subset A$, and let $f : A \longrightarrow \mathbb{R}$ be a multiplicative function satisfying the following conditions:*

- *f is strictly increasing on the set $P \cap A$.*
- *For each $u, v \in A$,*

$$\frac{f(uv)}{uv} \leq \frac{f(u)}{u} < 1. \tag{36}$$

- *For all primes $p, q \in A$ with $p \leq q$, we have,*

$$\begin{cases} \dfrac{q}{f(q)} \leq \dfrac{p}{f(p)}, \\ \dfrac{p}{f(p)} - \dfrac{q}{f(q)} \simeq 0. \end{cases} \tag{37}$$

Then there exists a finite set of positive integers $\{n_0, n_1, \ldots, n_m\} \subset W_\infty$ such that $p_r(f(n_i) + l) > p_{r-1}n_i$, for $i = 0, 1, \ldots, m$ with $m \simeq +\infty$.

Before presenting the proof of Proposition 4.1, the following lemma gives an example on the existence of the above function.

Lemma 4.2. *Let A be the subset of positive integers n for which any divisor d of n $(d \neq 1)$ is unlimited (i.e., if $n = q_1^{\alpha_1}q_2^{\alpha_2}\ldots q_s^{\alpha s}$, where $q_1, q_2, \ldots, q_s$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{N}$ are positive, then $n \in A$ if and only if $q_i \simeq +\infty$ for $i = 0, 1, \ldots, s$). For every $t \in \mathbb{N}^*$ limited, define the arithmetic function $\varphi_t : A \longrightarrow \mathbb{R}$ by,*

$$\varphi_t(n) = n\prod_{p|n}\left(1 - \frac{t}{p}\right).$$

Then (A, φ_t) satisfies the conditions of Proposition 4.1.

Proof. We prove this lemma as follows. By the way of contradiction assume that A is internal. Since $p_n \notin A$ for any standard n, we conclude from Cauchy's principle that $p_\omega \notin A$ for some unlimited integer ω. This is a contradiction. Further, by the definition of A we deduce that $W_\infty \subset A$.

Next, φ_t is multiplicative and strictly increasing on the set $P \cap A$ since $\varphi_t(uv) = \varphi_t(u)\varphi_t(v)$ for any $u, v \in A$ with $(u, v) = 1$ and $\varphi_t(p) < \varphi_t(q)$ for any primes $p, q \in A$ with $p < q$.

Let $u, v \in A$. Since $uv \in A$, it follows that,

$$\frac{\varphi_t(uv)}{uv} = \prod_{p|uv}\left(1 - \frac{t}{p}\right) \leq \prod_{p|u}\left(1 - \frac{t}{p}\right) = \frac{\varphi_t(u)}{u},$$

which gives,

$$\frac{\varphi_t(uv)}{uv} \leq \frac{\varphi_t(u)}{u} < 1.$$

Let $p, q \in A$ be two primes such that $p \leq q$. Then,

$$\begin{cases} \dfrac{q}{\varphi_t(q)} = \dfrac{q}{q-t} \leq \dfrac{p}{p-t} = \dfrac{p}{\varphi_t(p)}, \\ \dfrac{p}{f(p)} - \dfrac{q}{f(q)} = t\left(\dfrac{1}{p-t} - \dfrac{1}{q-t}\right) \simeq 0. \end{cases}$$

This proves the conditions stated in Proposition 4.1. □

Proof of Proposition 4.1. The proof is illustrated as follows. First, we prove that there exists a positive integer $n_0 \in W_\infty$ such that $p_r(f(n_0) + l) > p_{r-1}n_0$. Assuming the contrary, that is, for every $n \in W_\infty$ we have $p_r(f(n) + l) \leq p_{r-1}n$. By (36) and (37), there exits $p_m \in \mathbb{N}$ unlimited prime such that,

$$\begin{cases} \dfrac{p_m}{f(p_m)} = 1 + \phi, \\ \omega\phi \leq \dfrac{1}{p_{r-1}}, \end{cases}$$

for some $\omega \simeq +\infty$ and $\phi \simeq 0$. Define the set,

$$\left\{a \in \mathbb{N} \ : \ \left(\frac{p_m}{f(p_m)}\right)^a < \frac{p_{r-1}+1}{p_{r-1}}\right\}, \tag{38}$$

which is internal and containing $\mathbb{N}^\sigma$. From Cauchy's principle, there exists an unlimited integer a_0 such that,

$$\left(\frac{p_m}{f(p_m)}\right)^{a_0} < \frac{p_{r-1}+1}{p_{r-1}}.$$

Now, we set $n = p_m p_{m+1} \dots p_{m+a_0-1}$, that is, $n \in W_\infty$. By the first condition of (37) we obtain,

$$\frac{p_{m+i}}{f(p_{m+i})} > \frac{p_{m+i+1}}{f(p_{m+i+1})}, \quad \text{for } i = 0, 1, \dots, a_0 - 2.$$

Since f is multiplicative, it follows from the induction hypothesis that,

$$\frac{p_r}{p_{r-1}} \leq \frac{n}{f(n)+l} < \frac{n}{f(n)} = \prod_{i=0}^{a_0-1}\frac{p_{m+i}}{f(p_{m+i})} < \left(\frac{p_m}{f(p_m)}\right)^{a_0} < \frac{p_{r-1}+1}{p_{r-1}}, \tag{39}$$

Thus, from the first and the last inequality of (39) we get $p_r < p_{r-1} + 1$, which is absurd.

Finally, let $n_0 = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_m^{\alpha_m} \in W_\infty$ such that $p_r\left(f\left(n_0\right)+l\right) > p_{r-1}n_0$, where $q_1, q_2, \dots, q_m$ are prime numbers such that $q_1 < q_2 < \dots < q_m$, $m, q_1 \simeq +\infty$ and $\alpha_1, \alpha_2, \dots, \alpha_m$ are positive integers. If we put $n_i = \dfrac{n_0}{q_i}$ $(1 \leq i \leq m)$, then $n_i \in W_\infty$. Moreover, by (36) we get,

$$\frac{f\left(n_i\right)+l}{n_i} \geq \frac{f\left(n_i q_i\right)+l}{n_i q_i} > \frac{p_{r-1}}{p_r},$$

for $1 \leq i \leq m$. Thus, $p_r\left(f\left(n_i\right)+l\right) > p_{r-1}n_i$ for $1 \leq i \leq m$.

Proposition 4.1 is proved. □

In the following theorem we study the difference $\varphi(n) - 2^s \cdot p$ ([11]), where n is a composite unlimited integer, $s \geq 1$ is limited and $p \in P$ is unlimited.

Recall that *Fermat numbers* are known by their formula: $F_n = 2^{2^n} + 1$. The numbers F_0, F_1, F_2, F_3 and F_4 are the only known Fermat primes. Moreover, we do not know whether F_n is prime for some $n > 4$.

Theorem 4.5. *Let n be one of the natural numbers.*

(*i*) *An unlimited composite odd integer which is not divisible by any Fermat primes.*

(*ii*) *The product of certain Fermat primes and an unlimited composite odd integer which is not divisible by any Fermat primes.*

Then, $\varphi(n)$ is not of the form $2^s p$, where $s \in \mathbb{N}$ is limited and $p \in P$ is unlimited.

We prove the following lemma:

Lemma 4.3. *Let $n = \omega_1\omega_2$, where $\omega_1, \omega_2 \in \mathbb{N}$ are unlimited. Then, $\varphi(n)$ has the same form.*

Proof. From the definition of φ, we can immediately deduce that $\varphi(n)$ is unlimited if and only if n is unlimited. Therefore,

$$\varphi(n) = \varphi(\omega_1)\varphi(\omega_2)\frac{d}{\varphi\left(d\right)},$$

where $d = (\omega_1, \omega_2)$. We point out two cases:

Case 1: When d is limited. Since $\varphi\left(d\right)$divides $\varphi(\omega_1)$, $\varphi(n)$ is equal to the product of two unlimited integers: $\dfrac{\varphi(\omega_1)}{\varphi\left(d\right)}$ and $\varphi(\omega_2)d$.

Case 2: When d is unlimited. Since $\varphi\left(d\right)$divides $\varphi(\omega_2)$, $\varphi(n)$ is also equal to the product of two unlimited integers: $\varphi(\omega_1)$ and $\dfrac{\varphi(\omega_2)}{\varphi\left(d\right)}d$.

The proof is done. □

Proof of Theorem 4.5. We prove our theorem as follows:

Let n be unlimited such that $\varphi(n) = 2^s p$, where s is limited and p is unlimited prime number. From Lemma 4.3, n cannot be equal to the product of two unlimited positive integers. Thus, we must have $n = ap'$ with $a \geq 2$ is limited and p' is unlimited prime number. We study the two cases:

Case 1: Assume that n satisfies (i). Since $\varphi(n) = \varphi(ap') = \varphi(a)(p'-1) = 2^s p$ and $(p, \varphi(a)) = 1$, it follows that $(p'-1)$ is divisible by p. Thus, there exists $t \in \mathbb{N}$ such that $\varphi(a)t = 2^s$. That is, $\varphi(a)$ is a power of 2. On the other hand, by using the result stated in [13, Problem 533, p. 72], the integer a must be of the form $2^\nu F_0 F_1 \dots F_m$, where $\nu \geq 0$ and $F_j = 2^{2^j} + 1$ are Fermat primes for $j = 0, 1, \dots, m$. This means, the integer n is either even or divisible by some Fermat prime F_j with $j \geq 0$, contradicting the hypothesis.

Case 2: Assume that n satisfies (ii). If n is divisible by an unlimited Fermat prime, then $\varphi(n)$ is equal to the product of two unlimited integers, which is impossible. Therefore, $n = \prod_{j=0}^{m} F_j b$, where b satisfies (i) and $(F_j)_{0 \leq j \leq m}$ are limited Fermat primes for some $m \geq 0$. Since $(\prod_{j=0}^{m} F_j, b) = 1$, it follows that $\varphi(n)$ is of the form $2^s \varphi(b)$. By using the case (i), $\varphi(b)$ is not of the form $2^s p$. Therefore, it is the same for $\varphi(n)$. This is a contradiction as well.

Theorem 4.5 is now completely proved. □

5 Conclusion

The present study deals with Diophantine inequalities using positive integers having sufficiently large number of distinct prime factors. In fact, by using the Prime Number Theorem, Dirichlet's Theorem about primes in an arithmetic progression and Bertrand's theorem, we have proved in several cases that $p_r F(n) - p_{r-1} G(n)$ is both positive and negative infinitely often, where p_r denotes the r-th prime number, F and G are two number-theoretic functions and $n \in \mathbb{N}$ has a sufficiently large number of distinct prime factors. These number-theoretic functions are to be chosen multiplicative such as the Kernel and the Euler's function of the positive integer n.

As an application of nonstandard analysis, this work also includes some study using positive integers having unlimited number of distinct prime factors. More precisely, we have studied the same expression by using infinite external subsets of W_k. Indeed, we showed also that that $p_r \gamma^N(n) - p_{r-1} \gamma^N(n+l)$ and $p_r \varphi_s(n) - p_{r-1} n^s$ change sign on some infinite subsets $A \subset \mathbb{N}$.

For further research, we propose the following questions:

1. Recall that a powerful number is a positive integer n such that if a prime p divides n, then p^2 divides n. We ask if there is an infinite subset $A \subset W_k$ such that for any $n \in A$, one has:
$$\begin{cases} p_r \gamma^N(n) > p_{r-1} \gamma^N(n+l), & \text{if } n \text{ is powerful,} \\ p_r \gamma^N(n) < p_{r-1} \gamma^N(n+l), & \text{otherwise.} \end{cases}$$
2. Does $p_r \varphi_s(n) - p_{r-1} n^s$ change sign for infinitely many $n \in \overline{W}_\infty$?

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References

[1] R. Agnihotri & K. Chakraborty (2021). Sign changes of certain arithmetical function at prime powers. *Czechoslovak Mathematical Journal*, *71*(4), 1221–1228. https://doi.org/10.21136/CMJ.2021.0398-20.

[2] P. Borwein, S. K. Choi & H. Ganguli (2013). Sign changes of the Liouville function on quadratics. *Canadian Mathematical Bulletin*, *56*(2), 251–257. https://doi.org/10.4153/CMB-2011-166-9.

[3] P. Borwein, R. Ferguson & M. Mossinghoff (2008). Sign changes in sums of the Liouville function. *Mathematics of Computation*, *77*(263), 1681–1694. https://doi.org/10.1090/S0025-5718-08-02036-X.

[4] F. Diener & G. Reeb (1989). *Analyse Non Standard*. Hermann, Paris.

[5] F. Diener & M. Diener (2012). *Nonstandard Analysis In Practice*. Springer Science & Business Media, Heidelberg.

[6] B. Dinis & I. Van Den Berg (2019). *Neutrices and External Numbers: A Flexible Number System*. Chapman and Hall/CRC, Boca Raton.

[7] B. Djamel (2016). Notes on certain arithmetic inequalities involving two consecutive primes. *Malaysian Journal of Mathematical Sciences*, *10*(3), 253–268.

[8] B. Djamel & B. Abdelmadjid (2015). Non-classical study on the simultaneous rational approximation. *Malaysian Journal of Mathematical Sciences*, *9*(2), 209–225.

[9] R. Guy (2004). *Unsolved Problems in Number Theory* volume 1. Springer Science & Business Media, New York. https://doi.org/10.1007/978-0-387-26677-0.

[10] G. H. Hardy & J. E. Littlewood (1914). Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive. In *Proceedings of the London Mathematical Society*, volume 2 pp. 174–191. Oxford University Press, London. https://doi.org/10.1112/plms/s2-13.1.174.

[11] K. Hrbacek (2020). On factoring of unlimited integers. *Journal of Logic and Analysis*, *12*(5), 1–6. https://doi.org/10.4115/jla.2020.12.5.

[12] V. G. Kanoveĭ & M. Reeken (2004). *Nonstandard Analysis, Axiomatically*. Springer Science & Business Media, Heidelberg. https://doi.org/10.1007/978-3-662-08998-9.

[13] J. M. D. Koninck & A. Mercier (2007). *1001 Problems In Classical Number Theory*. American Mathematical Society, Providence.

[14] J. E. Littlewood (1914). Sur la distribution des nombres premiers. In *Comptes Rendus De l'Académie Des Sciences*, volume 158 pp. 1869–1872. L'Academie des Sciences, Paris.

[15] M. B. Nathanson (2000). *Elementary Methods in Number Theory* volume 195. Springer Science & Business Media, New York. https://doi.org/10.1007/b98870.

[16] E. Nelson (1977). Internal set theory: A new approach to nonstandard analysis. *Bulletin of the American Mathematical Society*, *83*(6), 1165–1198.

[17] J. Sándor (2002). *Geometric Theorems, Diophantine Equations, and Arithmetic Functions*. Infinite Study, Ann Arbor, Michigan.

[18] I. Van den Berg & V. Neves (2007). *The Strength of Nonstandard Analysis*. Springer Science & Business Media, Vienna. https://doi.org/10.1007/978-3-211-49905-4.

[19] M. Väth (2007). *Nonstandard Analysis*. Springer Science & Business Media, Basel. https://doi.org/10.1007/978-3-7643-7774-8.

[20] D. Wells (2005). *Prime Numbers: The Most Mysterious Figures in Math*. John Wiley & Sons, Hoboken, New Jersey.

[21] S. Y. Yan (2013). *Number Theory For Computing*. Springer Science & Business Media, Heidelberg. https://doi.org/10.1007/978-3-662-04773-6.